Lecture 02: Mathematical Basics (Probability)

- Sample Space: Ω is a set of outcomes (it can either be finite or infinite)
- Random Variable: $\mathbb{X}$ is a random variable that assigns probabilities to outcomes

Example: Let $\Omega = \{\text{Heads}, \text{Tails}\}$. Let $\mathbb{X}$ be a random variable that outputs Heads with probability $1/3$ and outputs Tails with probability $2/3$

- The probability that $\mathbb{X}$ assigns to the outcome x is represented by

$$\mathbb{P}\left[\mathbb{X} = x\right]$$

Example: In the ongoing example $\mathbb{P}\left[\mathbb{X} = \text{Heads}\right] = 1/3$.

Function of a Random Variable

- Let $f\colon \Omega \to \Omega'$ be a function
- Let $\mathbb{X}$ be a random variable over the sample space $\mathbb{X}$
- We define a new random variable $f(\mathbb{X})$ is over Ω' as follows

$$\mathbb{P}\left[f(\mathbb{X}) = y\right] = \sum_{x\in\Omega\colon\, f(x)=y} \mathbb{P}\left[\mathbb{X} = x\right]$$

Joint Distribution and Marginal Distributions I

- Suppose $(\mathbb{X}_1, \mathbb{X}_2)$ is a random variable over $\Omega_1 \times \Omega_2$.
  - Intuitively, the random variable $(\mathbb{X}_1, \mathbb{X}_2)$ takes values of the form (x_1, x_2), where the first coordinate lies in Ω_1, and the second coordinate lies in Ω_2

For example, let $(\mathbb{X}_1, \mathbb{X}_2)$ represent the temperatures of West Lafayette and Lafayette. Their sample space is $\mathbb{Z} \times \mathbb{Z}$. Note that these two outcomes can be correlated with each other.

Joint Distribution and Marginal Distributions II

- Let $P_1\colon \Omega_1 \times \Omega_2 \to \Omega_1$ be the function $P_1(x_1, x_2) = x_1$ (the projection operator)
- So, the random variable $P_1(\mathbb{X}_1, \mathbb{X}_2)$ is a probability distribution over the sample space Ω_1
- This is represented simply as $\mathbb{X}_1$, the marginal distribution of the first coordinate
- Similarly, we can define $\mathbb{X}_2$

Conditional Distribution

- Let $(\mathbb{X}_1, \mathbb{X}_2)$ be a joint distribution over the sample space $\Omega_1 \times \Omega_2$
- We can define the distribution $(\mathbb{X}_1 \mid \mathbb{X}_2 = x_2)$ as follows
  - This random variable is a distribution over the sample space Ω_1
  - The probability distribution is defined as follows

$$\mathbb{P}\left[\mathbb{X}_1 = x_1 \mid \mathbb{X}_2 = x_2\right] = \frac{\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right]}{\sum_{x \in \Omega_1} \mathbb{P}\left[\mathbb{X}_1 = x, \mathbb{X}_2 = x_2\right]}$$

For example, conditioned on the temperature at Lafayette being 0, what is the conditional probability distribution of the temperature in West Lafayette?

Bayes' Rule

Theorem (Bayes' Rule)

Let $(\mathbb{X}_1, \mathbb{X}_2)$ *be a joint distribution over the sample space* (Ω_1, Ω_2)*. Let* $x_1 \in \Omega_1$ *and* $x_2 \in \Omega_2$ *be such that* $\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right] > 0$*. Then, the following holds.*

$$\mathbb{P}\left[\mathbb{X}_1 = x_1 \mid \mathbb{X}_2 = x_2\right] = \frac{\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right]}{\mathbb{P}\left[\mathbb{X}_2 = x_2\right]}$$

The random variables $\mathbb{X}_1$ and $\mathbb{X}_2$ are independent of each other if the distribution $(\mathbb{X}_1 \mid \mathbb{X}_2 = x_2)$ is identical to the random variable $\mathbb{X}_1$, for all $x_2 \in \Omega_2$ such that $\mathbb{P}\left[\mathbb{X}_2 = x_2\right] > 0$

We can generalize the Bayes' Rule as follows.

Theorem (Chain Rule)

Let $(\mathbb{X}_1, \mathbb{X}_2, \ldots, \mathbb{X}_n)$ *be a joint distribution over the sample space* $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. *For any* $(x_1, \ldots, x_n) \in \Omega_1 \times \cdots \times \Omega_n$ *we have*

$$\mathbb{P}\left[\mathbb{X}_1 = x_1, \ldots, \mathbb{X}_n = x_n\right] = \prod_{i=1}^{n} \mathbb{P}\left[\mathbb{X}_i = x_i \mid \mathbb{X}_{i-1} = x_{i-1} \ldots, \mathbb{X}_1 = x_1\right]$$

Important: Why use Bayes' Rule I

In which context do we foresee to use the Bayes' Rule to compute joint probability?

- Sometimes, the problem at hand will clearly state how to sample $\mathbb{X}_1$ and then, conditioned on the fact that $\mathbb{X}_1 = x_1$, it will state how to sample $\mathbb{X}_2$. In such cases, we shall use the Bayes' rule to calculate

$$\mathbb{P}\left[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2\right] = \mathbb{P}\left[\mathbb{X}_1 = x_1\right]\mathbb{P}\left[\mathbb{X}_2 = x_2 | \mathbb{X}_1 = x_1\right]$$

- Let us consider an example.
  - Suppose $\mathbb{X}_1$ is a random variable over $\Omega_1 = \{0, 1\}$ such that $\mathbb{P}\left[X_1 = 0\right] = 1/2$. Next, the random variable $\mathbb{X}_2$ is over $\Omega_2 = \{0, 1\}$ such that $\mathbb{P}\left[X_2 = x_1 | \mathbb{X}_1 = x_1\right] = 2/3$. Note that $\mathbb{X}_2$ is biased towards the outcome of $\mathbb{X}_1$.
  - What is the probability that we get $\mathbb{P}\left[\mathbb{X}_1 = 0, \mathbb{X}_2 = 1\right]$?

Important: Why use Bayes' Rule II

- To compute this probability, we shall use the Bayes' rule.

$$\mathbb{P}\left[\mathbb{X}_1 = 0\right] = 1/2$$

Next, we know that

$$\mathbb{P}\left[\mathbb{X}_2 = 0 | \mathbb{X}_1 = 0\right] = 2/3$$

Therefore, we have $\mathbb{P}\left[\mathbb{X}_2 = 1 | \mathbb{X}_1 = 0\right] = 1/3$. So, we get

$$\begin{aligned} \mathbb{P}\left[\mathbb{X}_1 = 0, \mathbb{X}_2 = 1\right] &= \mathbb{P}\left[\mathbb{X}_1 = 0\right] \mathbb{P}\left[\mathbb{X}_2 = 1 | \mathbb{X}_1 = 0\right] \\ &= (1/2) \cdot (1/3) = 1/6 \end{aligned}$$

Independence of Random Variables

- Consider a joint distribution $(\mathbb{X}_1, \mathbb{X}_2)$ over the sample space $\Omega_1 \times \Omega_2$
- The marginal distributions $\mathbb{X}_1$ and $\mathbb{X}_2$ are independent of each other, if for all $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ we have: If $\mathbb{P}[\mathbb{X}_1 = x_1] > 0$ then

$$\mathbb{P}[\mathbb{X}_2 = x_2] = \mathbb{P}\left[\mathbb{X}_2 = x_2 | \mathbb{X}_1 = x_1\right].$$

- Equivalently, the following condition is satisfied

$$\mathbb{P}[\mathbb{X}_1 = x_1] \cdot \mathbb{P}[\mathbb{X}_2 = x_2] = \mathbb{P}[\mathbb{X}_1 = x_1, \mathbb{X}_2 = x_2].$$

Probability: First Example I

- Let $\mathbb{S}$ be the random variable representing whether I studied for my exam. This random variable has sample space $\Omega_1 = \{\mathsf{Y}, \mathsf{N}\}$
- Let $\mathbb{P}$ be the random variable representing whether I passed my exam This random variable has sample space $\Omega_2 = \{\mathsf{Y}, \mathsf{N}\}$
- Our sample space is $\Omega = \Omega_1 \times \Omega_2$
- The joint distribution $(\mathbb{S}, \mathbb{P})$ is represented in the next page

Probability: First Example II

s	p	$\mathbb{P}[\mathbb{S} = s, \mathbb{P} = p]$
Y	Y	1/2
Y	N	1/4
N	Y	0
N	N	1/4

Probability: First Example III

Here are some interesting probability computations
The probability that I pass.

$$\begin{aligned}\mathbb{P}\left[\mathbb{P}=\mathsf{Y}\right] &= \mathbb{P}\left[\mathbb{S}=\mathsf{Y},\mathbb{P}=\mathsf{Y}\right]+\mathbb{P}\left[\mathbb{S}=\mathsf{N},\mathbb{P}=\mathsf{Y}\right]\\ &= 1/2+0=1/2\end{aligned}$$

Probability: First Example IV

The probability that I study.

$$\begin{aligned}\mathbb{P}\left[\mathbb{S} = \mathsf{Y}\right] &= \mathbb{P}\left[\mathbb{S} = \mathsf{Y}, \mathbb{P} = \mathsf{Y}\right] + \mathbb{P}\left[\mathbb{S} = \mathsf{Y}, \mathbb{P} = \mathsf{N}\right] \\ &= 1/2 + 1/4 = 3/4\end{aligned}$$

The probability that I pass conditioned on the fact that I studied.

$$\mathbb{P}\left[\mathbb{P} = \mathsf{Y} \mid \mathbb{S} = \mathsf{Y}\right] = \frac{\mathbb{P}\left[\mathbb{P} = \mathsf{Y}, \mathbb{S} = \mathsf{Y}\right]}{\mathbb{P}\left[\mathbb{S} = \mathsf{Y}\right]}$$

$$= \frac{1/2}{3/4} = \frac{2}{3}$$

Probability: Second Example I

- Let $\mathbb{T}$ be the time of the day that I wake up. The random variable $\mathbb{T}$ has sample space $\Omega_1 = \{4, 5, 6, 7, 8, 9, 10\}$
- Let $\mathbb{B}$ represent whether I have breakfast or not. The random variable $\mathbb{B}$ has sample space $\Omega_2 = \{T, F\}$
- Our sample space is $\Omega = \Omega_1 \times \Omega_2$
- The joint distribution of $(\mathbb{T}, \mathbb{B})$ is presented on the next page

t	b	$\mathbb{P}[\mathbb{T} = t, \mathbb{B} = b]$
4	T	0.03
4	F	0
5	T	0.02
5	F	0
6	T	0.30
6	F	0.05
7	T	0.20
7	F	0.10
8	T	0.10
8	F	0.08
9	T	0.05
9	F	0.05
10	T	0
10	F	0.02

Probability: Second Example III

- What is the probability that I have breakfast conditioned on the fact that I wake up at or before 7?

Formally, what is $\mathbb{P}\left[\mathbb{B} = \mathsf{T} \mid \mathbb{T} \leqslant 7\right]$?